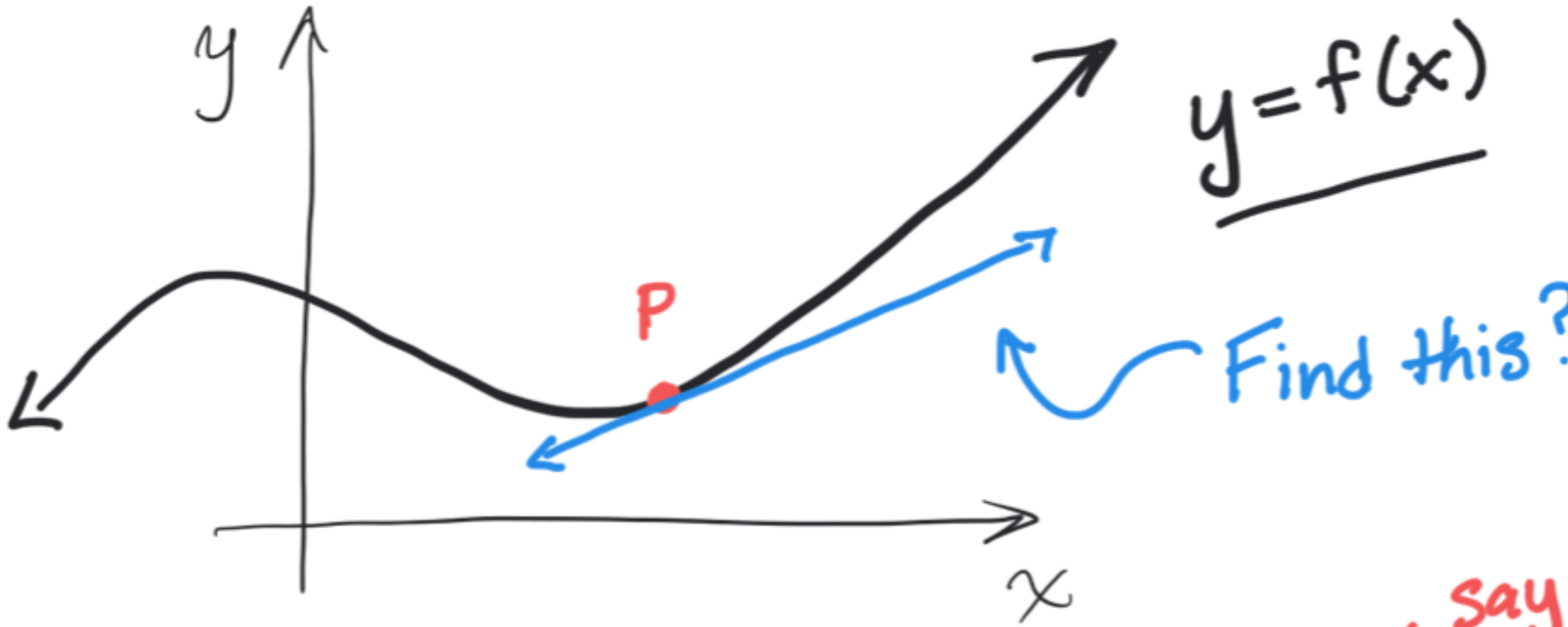
Lecture Notes 2-1: The Tangent and Velocity Problems

The importance of a good question.

Question 1: Given the graph of a function $y = f(x)$ and a point P on this graph, how do you *define* and *find* the equation of the tangent line to the graph at P?



say in feet

Question 2: Given the position of an object (say a cell phone) at any time, how do you *define* and *find* the velocity of the object at a particular instant (say the moment your child launches it off a cliff)?

That is, if I know an object's position at any time, can I find its velocity at an instant in time?

say in seconds

Say in feet/sec.

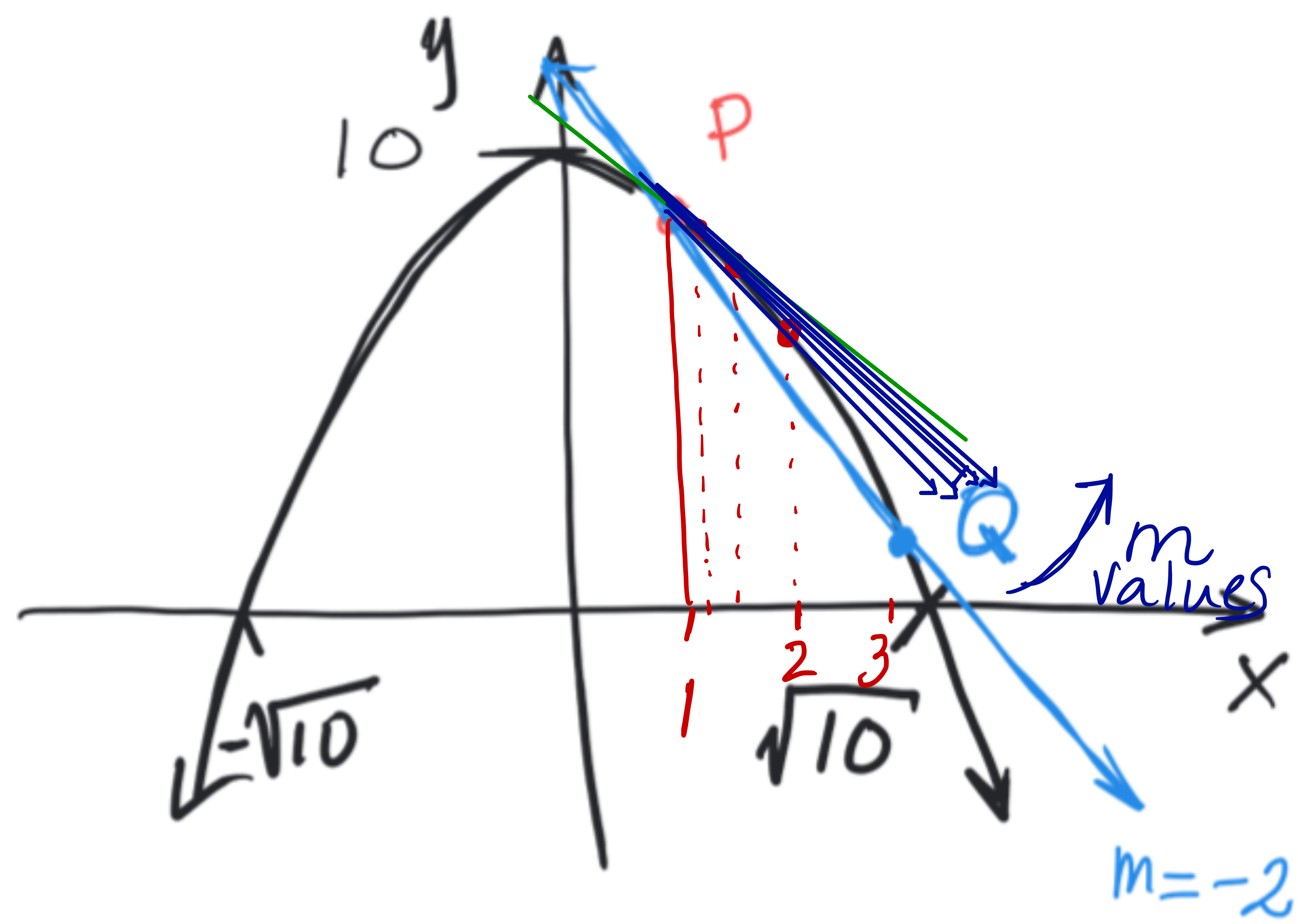
Some Facts:

- These questions are old. (200BC or older depending on your interpretation)
- These questions are hard, taking more than a thousand years and untold numbers of mathematicians to answer.
- Before finding solid mathematical ground, some of its ideas were even more controversial than Donald Trump's tweets are today!
- Attempts to answer these two questions is *part* of what led to the development/discovery of Calculus.
- The ideas you learn in calculus explain planetary motion or where a projectile will land or predict how fast an infection will spread.
- **Most importantly and perhaps obviously,** *the questions that motivated the development of calculus go a long way to explaining the definitions and applications we see later*

EXAMPLE 1: Let $f(x) = (10 - x^2)/2$.

1. Sketch a LARGE graph of $f(x)$ in the space to the right. Include any x- or y-intercepts.

(parabola, opening down)



2. Let P be the point on the curve where $x = 1$ and let Q be the point on the curve where $x = 3$. Find the y-coordinate for P and Q and plot them on your graph above.

X	1	3
y	9/2 (=4.5)	1/2

P(1, 4.5)

Q(3, 0.5)

3. DEFINITION: A *secant line* on a graph is simply the line determined by two points on the graph. Find the EQUATION of the secant line determined by the points P and Q and graph it above.

$$m = \frac{4.5 - 0.5}{1 - 3} = \frac{4}{-2} = -2 \text{ (slope)}$$

equation of Secant

$$y - 4.5 = -2(x-1)$$
$$= -2x + 2$$

$$\boxed{y = -2x + 6.5}$$

4. Label the line you just plotted above with its *slope*.

5. For the FIVE points Q_1, Q_2, Q_3, Q_4, Q_5 with x-coordinates $2,\ 1.5,\ 1.25,\ 1.125, 1.0625$, find the y-coordinate, plot the point, plot the secant line determined by P and Q_i, and label the line with its slope.

X	2	1.5	1.25	1.125	1.0625
y	3	3.875	4.219	4.367	4.4355
m slope $\frac{y-4.5}{x-1}$	-1.5	-1.25	-1.125	-1.0625	-1.0325

m values

It's in green

6. Sketch what YOU think the tangent to $f(x)$ at the point P should look like...???

7. What do you observe about the relationship between the secant lines you **calculated** and the tangent line you **guessed at**?

The SLOPES of the secant lines gets closer to the SLOPE of the tangent.

8. What is the significance of the **words in bold** in the previous question?

While we want the slope of the tangent, what we can calculate (find precisely) are slopes of secants.

9. What PART of the tangent line is indicated by the sequence of secant lines? SLOPE

10. Write the *equation* of the tangent line to $f(x)$ at P. Does this answer seem reasonable? Why or why not?

point $(1, 4.5)$

slope $m = -1$ (guess!?)

$y - 4.5 = -1(x-1)$

$y = -x + 1 + 4.5$

$\boxed{y = 5.5 - x}$

11. In *plain old ENGLISH SENTENCES* how would you explain (step-by-step) how to find the *equation* of the tangent line?

1. Find points on curve getting closer to point P.
2. Find slopes of secants determined by P and points from #1.
3. Look at what value the slopes from #2 get closer to. Use this for the value of m.
4. Write equation using P and m.

12. In the previous exercise, we chose points (Q_i's) on the *right* of the point P, what would happen if we had chosen points on the *left*? They will also have slopes getting closer to -1.

EXAMPLE 2

t (seconds)	0	1	2	3	4	5	6
s (feet)	0	4.9	20.6	46.5	79.2	124.8	176.7

The table above shows the position of a motorcyclist after accelerating from rest.

$$\text{average velocity} = \frac{\Delta \text{ position}}{\Delta \text{ time}}$$

1. Find the average velocity for each time period and include units in your answer.

   (a) From $t = 2$ to $t = 4$. $\rightarrow$ $\text{average velocity} = \dfrac{79.2 - 20.6}{4-2} = 29.3 \text{ ft/s}$

   (b) From $t = 3$ to $t = 4$. $\rightarrow$ $\dfrac{79.2 - 46.5}{4-3} = 32.7 \text{ ft/s}$

   (c) From $t = 4$ to $t = 5$. $\rightarrow$ $\dfrac{124.8 - 79.2}{5-4} = 45.6 \text{ ft/s}$

   (d) From $t = 4$ to $t = 6$. $\rightarrow$ $\dfrac{176.7 - 79.2}{6-2} = 48.75 \text{ ft/s}$

2. In words in English, what should the *average velocity* of an object be?

   Given a $\boxed{\text{time interval}}$, find : $\dfrac{\text{change in position}}{\text{change in time}}$

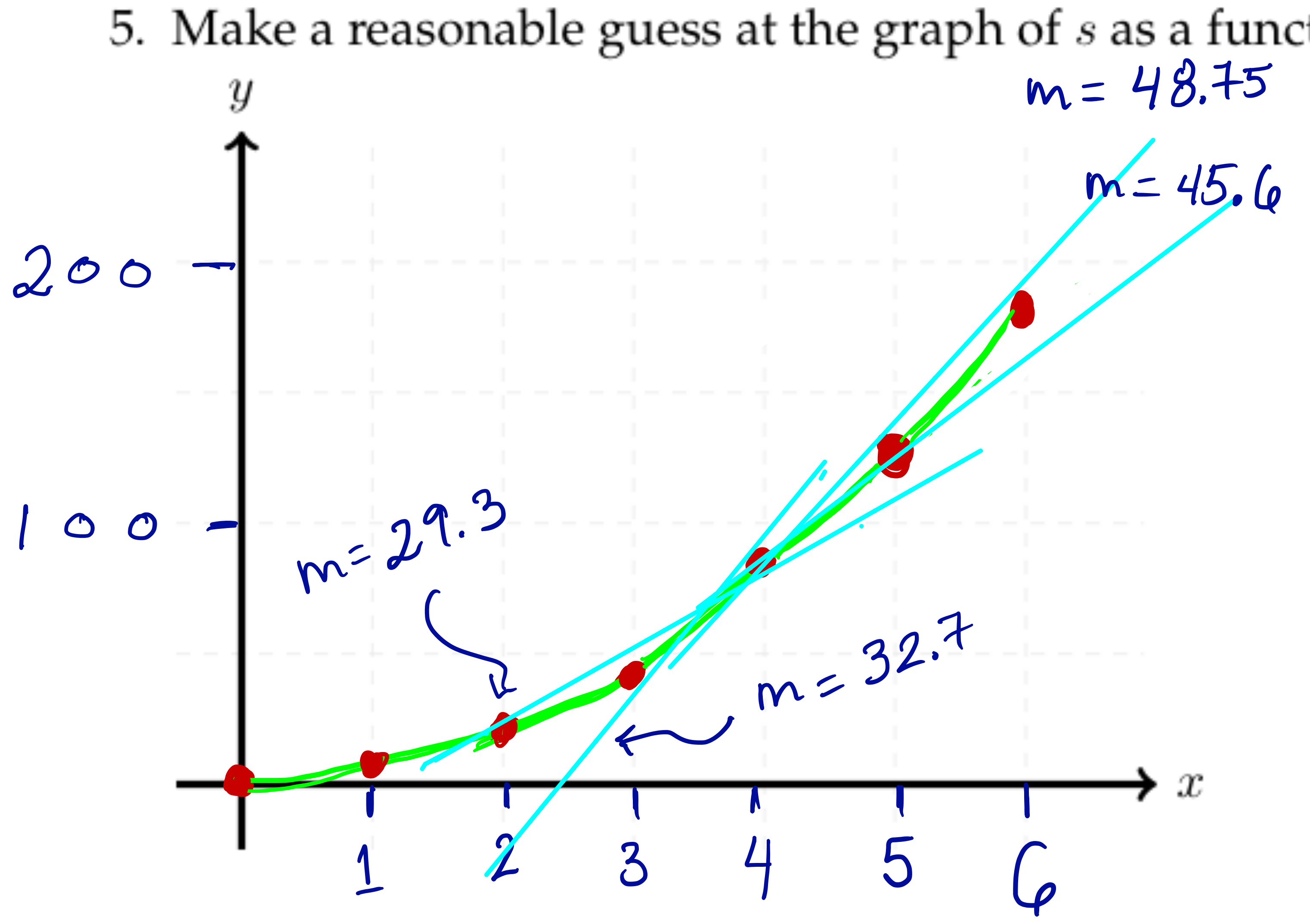
3. In words in English, what should the *instantaneous velocity* of an object be?

   Use a super-small time interval and find the average velocity on this super-small interval.

4. If the object is a motorcycle, what should the *instantaneous velocity* of an object be?

   What the speedometer shows.

5. Make a reasonable guess at the graph of s as a function of t using the data from the table.



Secant lines

in blue

Slopes in dk blue

I chose to average the average velocities in the second before + the second after.

6. Estimate the instantaneous velocity of the motorcycle four seconds after accelerating from rest. Is your answer reasonable?

	2 sec	1 sec		1 sec	2 sec
time interval	2 to 4	3 to 4	4	4 to 5	4 to 6
avg. velocity	29.3	32.7	?	45.6	48.75

Jill's guess

$\frac{45.6 + 32.7}{2} = 39.15$ ft/sec

instantaneous velocity, a time interval of... zero??

7. On the graph in part 5, DRAW and LABLE all of the calculations from parts a-d in question 1.

8. State explicitly and in complete sentences in English the relationship between the tangent problem and the velocity problem.

Solving the tangent problem means finding the slope of a given curve. Solving the velocity problem means finding instantaneous velocity given position of object at given times.

The slope of tangent is the same as instantaneous velocity when the curve represents (or models) the position of an object over time. In both cases we approximate the slope or instantaneous velocity using secant lines or average velocity

The Last Word(s): In real estate, the three most important things to keep in mind are (1) location, (2) location, (3) location.
As of today, you now know the three most important techniques in calculus:

approximate, approximate, approximate !!